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A Computational Method of Solving Free-Boundary Problems in Vortex Dynamics*

ALEXANDER EYDELAND AND BRUCE TURKINGTON

Department of Mathematics and Statistics
University of Massachusetts, Amherst, Massachusetts 01003

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A general method for computing two-dimensional and axisymmetric three-dimensional solutions of the steady Euler equations is developed. The method is based upon variational principles which are formulated entirely in terms of the natural conserved quantities for the unsteady Euler equations. The vorticity fields considered have a free-boundary and may be defined either in a domain with boundary or in free space; moreover, the common restriction to the case of constant vorticity regions is not imposed. The global convergence of the resulting iterative solver is established. The numerical implementation of the method is discussed and several computational examples are given. © 1988 Academic Press, Inc.

INTRODUCTION

In this article we present a general numerical method for computing steady vortex flows in an ideal fluid (that is, an incompressible, inviscid fluid governed by the Euler equations). We consider only flows that are either two-dimensional or three-dimensional and axisymmetric (possibly with swirl). In this context, we use the term "steady vortex flow" to mean a steady flow induced by one or several vortices, each being a region of one-signed vorticity, embedded in an irrotational ambient flow. The determination of such a flow evidently involves the solution of a nonlinear, free-boundary problem: the boundary of the vortex region is required to be a streamline across which the tangential components of velocity are continuous, and the vorticity-streamfunction dependence is specified within the vortex region. The standard literature discusses various specific flows of this type (see, for instance, [3, 15]), including such classical examples as steadily translating vortex pairs and rings, Föppl vortex wakes, and Von Karman vortex streets. Usually, numerical as well as analytical treatments of these flows rely upon certain simplifying assumptions, such as special symmetries among the vortices, particular flow geometries, or restrictive vorticity distributions (commonly "vortex patches"). Our aim in the present work, therefore, is to address the problem of computing steady

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vortex flows in a *general* setting, and to present the formulation, analysis, and implementation of our computational method in a coherent and unified fashion.

The method that we devise is a globally convergent iterative procedure, the particular form of which is dictated by the variational structure of the problem it is designed to solve. Indeed, our approach rests fundamentally on a general variational principle due to Arnol'd [1, 2] which characterizes steady flows in an ideal fluid as constrained extremals for the kinetic energy functional defined on an appropriate class of competing vorticities. From a computational point of view the utility of this variational approach is threefold. First, it leads to a formulation of the general mathematical problem entirely in terms of the natural physical invariants associated with the equations governing vortex dynamics—namely, energy, impulse (or momentum), enstrophy, and helicity. Second, it allows a rigorous analysis of the convergence properties of the iterative procedure, which is based essentially on the underlying variational structure. Third, it results in numerical algorithms that are widely applicable and yet quite easily implemented. Our method seems to be especially well suited to the computation of flows in general geometries and with general vorticity-streamfunction dependences. In this regard we note that our variational formulation differs slightly from that given by Arnol'd [1, 2], whose principle is based upon the so-called isovortical variations, as we prefer to prescribe the functional relation between vorticity and streamfunction instead.

The variational theory of steady vortex flows which constitutes the analytical counterpart of our work has been developed in a series of papers by the second author [14, 19, 20, 22] for both two-dimensional and three-dimensional axisymmetric flows. The conceptual basis of this theory has been furnished by Benjamin [4, 5], whose work is especially noteworthy because it connects the abstract variational principles with concrete model problems in ideal fluid dynamics. In [12] we have investigated another problem suggested in [5] concerning solitary planetary (or Rossby) waves in a zonal current on the beta-plane using a method very similar to that used in the present paper.

An alternate variational approach based directly on the semi-linear elliptic equation satisfied by the streamfunction has been employed by Fraenkel and Berger [13], Norbury [16], and others to establish existence theorems for vortex pairs and rings. Berestycki, Fernandez–Cara, and Glowinski [7] have examined the corresponding numerical results in considerable detail, utilizing an iterative procedure appropriate to this formulation. The variational principle undelying all of these results can be viewed as dual (in the sense of convex analysis) to our principle. Of course, there are also many other computational studies of steady flows in vortex dynamics which do not involve variational methods, and these results are referenced throughout the sequel when specific examples are discussed.

We restrict our attention in this paper to flows which either contain only a single vortex region or may be reduced to such by symmetry. Our methods, however, apply just as well to flows containing several vortices. In a subsequent paper we intend to extend the results given herein to systems of vortices in asymmetric configurations.

The article is organized as follows. In Section 1, we explain the formulation of the constrained maximization problems (with one or two linear constraints) in a general context, and we review some specific examples that supply relevant model problems. In Section 2, we first define the iterative procedure appropriate to each variational problem, and we then analyze the convergence properties of these iterative procedures. In Section 3, we discuss the implementation of the numerical method, and we exhibit the results of computations for vortex pairs and Föppl vortex wakes, as illustrative examples of the application of the general method.

1. Variational Principles for Steady Flows

In this section we formulate the variational problems that form the basis of our subsequent analysis. In 1A we recall the equations governing vortex dynamics in two dimensions, and we pose the steady flow problems in a general setting. We then present in 1B several specific flow problems of physical interest which motivate our general analysis and which serve as examples in our implemented computations. We give some further extensions to axisymmetric flows (with or without swirl) in three dimensions in 1C.

1A. *Constrained Maximization Problems*

Let $D \subseteq \mathbb{R}^2$ be a bounded, simply-connected domain with piecewise smooth boundary, ∂D, in the (x, y)-plane; let ν denote the unit outward normal on ∂D. We shall formulate the Euler equations for an ideal fluid flow in D in the vorticity/streamfunction form, writing $\omega = \omega(x, y, t)$ and $\psi = \psi(x, y, t)$ for the vorticity and streamfunction, respectively. Let G denote the Green operator which defines $\psi = G\omega$ as the solution of the problem:

$$-\Delta\psi = \omega \text{ in } D, \qquad \psi = 0 \text{ on } \partial D. \tag{1.1}$$

The governing equation of vortex dynamics is expressible as (see [3, 5, 23])

$$\omega_t + \frac{\partial(\omega, G\omega + \bar{\psi})}{\partial(x, y)} = 0 \qquad \text{in} \quad D \times (0, T), \tag{1.2}$$

where $\bar{\psi} = \bar{\psi}(x, y)$ is the streamfunction for an ambient (steady) irrotational flow in D upon which the vortex flow defined by ω is imposed; thus, $\Delta\bar{\psi} = 0$ in D and $\bar{\psi}$ is specified on ∂D, which amounts to specifying the flux at each point on the boundary.

We formulate the variational principles for steady vortex flows by following the dynamically natural approach of Arnol'd [1, 2], which consists in extremizing the kinetic energy over all isovortical variations of a given (extremal) flow. In the present context, this general approach is as follows. Let the kinetic energy functional be expressed as

$$E(\omega) = \iint_D \left[\tfrac{1}{2}\omega G\omega + \bar{\psi}\omega\right] dx\, dy. \tag{1.3}$$

For a given vorticity $\omega = \omega(x, y)$ we consider the variations $\tilde{\omega} = \tilde{\omega}(x, y; \tau)$ defined by solving the equation (in a neighborhood of $\tau = 0$);

$$\tilde{\omega}_\tau + \frac{\partial(\tilde{\omega}, \phi)}{\partial(x, y)} = 0 \text{ in } D \times (-\delta, \delta), \qquad \tilde{\omega}|_{\tau=0} = \omega,$$

given an arbitrary test function ϕ in D (ϕ is smooth in $\bar{D}$ and vanishes on ∂D). We find, after some calculation, that

$$\frac{d}{d\tau} E(\tilde{\omega})|_{\tau=0} = \iint_D \omega \frac{\partial(G\omega + \bar{\psi}, \phi)}{\partial(x, y)}\, dx\, dy.$$

Thus, the condition that ω be an extremal for E over all such variations is precisely the weak form of the dynamical equation (1.2) for steady solutions—namely,

$$0 = \iint_D \omega \frac{\partial(\phi, G\omega + \bar{\psi})}{\partial(x, y)}\, dx\, dy \qquad \text{for all test functions } \phi. \tag{1.4}$$

Of course, (1.4) simply expresses the requirement that the vorticity ω be constant along all of the streamlines associated with the streamfunction $G\omega + \bar{\psi}$. We therefore seek steady solutions for which this required vorticity-streamfunction dependence is specified explicitly in the form

$$\omega = \lambda f_s(G\omega + \bar{\psi}), \tag{1.5}$$

where λ is a vorticity strength parameter and $f_s(s)$ is a vorticity profile function. It is readily checked that (1.5) implies (1.4) for any (positive) constant λ and any (suitably smooth) real function $f(s)$.

Some restrictions on the above form are necessary in order to treat steady vortex flows with free-boundaries—that is, solutions $\omega = \omega(x, y)$ for which $\omega > 0$ in a sub-domain $\Omega \subset D$ and $\omega = 0$ in $D \backslash \Omega$—and these are the flows that hold our attention here. For this purpose we require that the specified function $f(s)$ is (1.5) satisfies the structure conditions:

$$\begin{aligned} &f \in C^2[0, +\infty), \qquad f(0) = f_s(0) = 0, \qquad f_{ss}(s) > 0 \text{ for } s > 0 \\ &\frac{ms^r}{r} \leqslant f(s) \leqslant \frac{Ms^r}{r} \qquad \text{for some} \quad 1 < r < \infty, 0 < m \leqslant M < \infty. \end{aligned} \tag{1.6}$$

Let $f^*(\sigma)$ be the conjugate function to $f(s)$ defined by the familiar formula

$$f^*(\sigma) = \sup_s [s\sigma - f(s)]; \tag{1.7}$$

then $f^*(\sigma) = sf_s(s) - f(s)$ and $f_{ss}(s) = [f^*_{\sigma\sigma}(\sigma)]^{-1}$ with $\sigma = f_s(s)$ or, equivalently, $s = f^*_\sigma(\sigma)$, as is easily verified. The special case of a power function $f(s) = s^r/r$ clearly furnishes the prototype upon which the general class defined in (1.6) is fashioned; in this case, $f^*(\sigma) = \sigma^{r'}/r'$ with $1/r + 1/r' = 1$. (With this prototypical case in mind, we shall emphasize the role of the exponent r in our nomenclature for the variational problems and iterative procedures introduced in the sequel, such as "Problem P_1^r," "Procedure Π_1^r," etc.) We now construct a variational problem whose solutions $\omega = \omega_\lambda$ satisfy (1.5) and are parametrized by λ for a fixed $f(s)$. For $\lambda > 0$, let the objective functional $\Phi_\lambda(\omega)$, representing energy modified by a certain generalized enstrophy integral, and the constraint functional, $C(\omega)$, representing circulation, be defined by

$$\Phi_\lambda(\omega) = E(\omega) - \iint_D \lambda f^*(\omega/\lambda)\, dx\, dy \tag{1.8}$$

$$C(\omega) = \iint_D \omega\, dx\, dy. \tag{1.9}$$

We consider the constrained maximization problem (characterizing steady vortex flows with free-boundaries).

PROBLEM P_1^r. Maximize $\Phi_\lambda(\omega)$ subject to $\omega \geqslant 0$ in D, $C(\omega) = 1$.

The normalization of the circulation constraint to unity is accomplished by rescaling both ω and λ. That a maximizer $\omega = \omega_\lambda$ for the problem P_1^r yields a steady flow satisfying (1.5) (and hence (1.4)) follows in the standard manner. Indeed, the variational conditions for P_1^r are

$$\begin{aligned} G\omega + \bar{\psi} - \mu &= f^*_\sigma(\omega/\lambda) \qquad \text{on} \quad \{\omega > 0\} \\ G\omega + \bar{\psi} - \mu &\leqslant 0 \qquad\qquad\quad \text{on} \quad \{\omega = 0\}, \end{aligned} \tag{1.10}$$

where μ is the Lagrange multiplier for the constraint $C(\omega) = 1$. It then follows, using the properties of the conjugate function, f^*, that

$$\omega = \lambda f_s((G\omega + \bar{\psi} - \mu)^+) \qquad (\phi^+ = \max\{\phi, 0\}), \tag{1.11}$$

the appropriate variant of (1.5). The vortex core $\Omega = \{G\omega + \bar{\psi} > \mu\}$ is, at least typically, a compactly contained subdomain of the fluid domain, and the solution given by (1.11) defines a steady flow whose velocities are continuous across the free-boundary, $\partial\Omega$.

An important case not included in the above discussion occurs when the solution has the form $\omega = \lambda\chi_\Omega$, where χ_Ω denotes the characteristic function for the set Ω. The appropriate constrained maximization problem in this case is

PROBLEM P_1^1. Maximize $E(\omega)$ subject to $0 \leqslant \omega \leqslant \lambda$ in D, $C(\omega) = 1$.

Now it is immediate that such a maximizer $\omega = \omega_\lambda$ is a steady solution of (1.4) since these constraints are invariant under the variations $\tilde{\omega}$ used to derive (1.4). As is shown in [19] the variational conditions for P_1^1 lead to the relation

$$\omega = \lambda\chi_{\{G\omega + \bar{\psi} - \mu > 0\}}, \tag{1.12}$$

where μ is the Lagrange multiplier as before; here it is assumed that $\lambda > 1/\text{meas}(D)$. Problem P_1^1 may be viewed as a limit of the problem P_1^r with $f(s) = s^r/r$ as $r \to 1$; the functional Φ_λ then represents a penalization of the functional E, which enforces the pointwise constraint $\omega/\lambda \leqslant 1$ in D as $r \to 1$. Indeed, we see that as $r \to 1$ the solutions given by (1.11) tend to the solution given by (1.12), since $f_s(s) = (s^+)^{r-1} \to \chi_{\{s>0\}}$. In the literature, a steady flow of the form (1.12) is often referred to as a "vortex patch."

We now pose alternate versions of P_1^r and P_1^1, in which the ambient flow (with streamfunction $\bar{\psi}$) is not specified directly but rather is derived as the result of an additional constraint. These variants arise quite naturally in certain physical applications (see 1B). Let a (normalized) streamfunction $\eta = \eta(x, y)$ be fixed with $\Delta\eta = 0$ in D, and let the corresponding generalized linear impulse (or momentum) be defined by

$$I(\omega) = \iint_D \eta\omega \, dx \, dy; \tag{1.13}$$

it may be assumed that η is specified (on ∂D) so that $\eta \geqslant 0$ on $\bar{D}$. Let the objective functionals Φ_λ and E be defined as above except that now the terms involving $\bar{\psi}$ are dropped. We consider the alternate problems having two linear constraints:

PROBLEM P_2^r. Maximize $\Phi_\lambda(\omega)$ subject to $\omega \geqslant 0$ in D, $C(\omega) = 1$, $I(\omega) = m$.

PROBLEM P_2^1. Maximize $E(\omega)$ subject to $0 \leqslant \omega \leqslant \lambda$ in D, $C(\omega) = 1$, $I(\omega) = m$,

where λ and m are prescribed positive parameters. (The interpretation of and motivation for the impulse constraint is given below in the discussion of the specific examples.) The variational conditions satisfied by a maximizer $\omega = \omega_{\lambda,m}$ for P_2^r or P_2^1 are, respectively,

$$\omega = \lambda f_s((G\omega - c\eta - \mu)^+) \tag{1.14}$$

$$\omega = \lambda\chi_{\{G\omega - c\eta - \mu > 0\}} \tag{1.15}$$

for Lagrange multipliers μ and c corresponding to the constraints $C(\omega)=1$ and $I(\omega)=m$, respectively. Clearly, these relations yield the effective ambient flow $\bar{\psi}=-c\eta$, when compared with the analogous relations (1.11) and (1.12).

The existence of a maximizer for each of the four problems stated above can be demonstrated quite easily using direct variational methods (see 1B for some references in special cases). Alternatively, the analysis of the iterative procedure given in Section 2 furnishes a constructive proof of existence of solutions. The variational conditions derived above as well as certain regularity properties of the solutions can also be established; this requires a penalization procedure for P_1^1 and P_2^1.

1B. *Specific Flow Geometries*

We summarize here several important special cases of the general problems formulated above. Although these examples do not constitute an exhaustive list of the possible applications, they do suffice to indicate the wide applicability of the variational method.

Model Problem

The simplest example concerns vortex flows in a bounded domain $D \subseteq \mathbb{R}^2$ with $\bar{\psi}=0$ in D (trivial ambient flow). The existence theory for P_1^r is available in [6] (where the equivalent so-called plasma problem is studied), and the analogous theory for P_1^1 is given in detail in [19]. The behavior of the solutions ω_λ as $\lambda \to +\infty$ is readily treated using the particular formulation of P_1^r and P_1^1 given here. Indeed, the main results in [19] address this question and establish, in particular, that ω_λ tends to a unit delta measure (a point vortex) located at an equilibrium point of the Kirchhoff–Routh Hamiltonian for the domain D (which is constructed from the Green function for D).

Vortex Pair Problem

This problem is actually posed on the half-plane $\{y>0\}$, but for our purposes (which concern numerical solutions) it is appropriate to consider the truncated version of the problem on a finite rectangle $D_{a,b}=\{|x|<a,\ 0<y<b\}$. A solution represents an opposite-signed, symmetric vortex pair translating with speed c in the x-direction. Thus, in the steady flow problem in $D_{a,b}$ the ambient streamfunction $\bar{\psi}=-cy$ defines a uniform flow $(-c, 0)$. In the alternate problems P_2^r and P_2^1, it is natural to define $\eta=y$, and so the functional $I(\omega)=\iint y\omega\,dx\,dy$ coincides with the x-component of the classical linear impulse (or momentum) (see [3]). The existence theory for P_1^1 and P_2^1 is carried out in this context in [20] (where $a, b=+\infty$ is also considered), and similar results for P_1^r and P_2^r $(1<r<+\infty)$ are easily obtained with the same methods. As $\lambda \to +\infty$, the solutions for P_2^r and P_2^1 tend to the point vortex pair at $(x, y)=(0, \pm m)$, as is shown in [20]. Further references concerning vortex pairs include [16, 17, 24].

Föppl Vortex Wake Problem

This problem is a variant of the preceding one in which an obstacle $\tilde{D}$ is now introduced into the (uniform) stream. We assume that $\tilde{D}$ is bounded with smooth $\partial\tilde{D}$ and is symmetrical about the x-axis; the fluid domain is therefore, $D_{a,b} = \{|x|<a,\ 0<y<b\}\backslash\tilde{D}$. The ambient flow is now the classical irrotational flow past $\tilde{D}$ with uniform velocity at ∞, say $(-c, 0)$; thus, we define $\psi = -c\eta$ where η satisfies

$$\Delta\eta = 0 \text{ in } D_{a,b}, \qquad \eta = 0 \text{ on } \partial\tilde{D}, \qquad \eta = y \text{ on } \partial D_{a,b}\backslash\partial\tilde{D}. \tag{1.16}$$

The interpretation of the impulse functional $I(\omega)$ is not standard in the presence of both vorticity and an obstacle. In [20, Appendix] it is shown that the (x-component of the) total linear impulse and the total energy are given by, respectively,

$$I^* = I(\omega) + Mc, \qquad E^* = E(\omega) + \tfrac{1}{2}Mc^2,$$

where $M = \iint_{D_{a,b}} |\nabla(\eta - y)|^2\, dx\, dy$ is the so-called induced mass of the obstacle $\tilde{D}$. Consequently, the functionals I and E may be interpreted as the vortex parts of the impulse and energy, while the terms Mc and $\frac{1}{2}Mc^2$ have standard interpretations in the classical theory of irrotational flows. (As is also demonstrated in [20, Appendix], the time derivative of I^* for an evolving vortex flow in $\mathbb{R}^2\backslash\tilde{D}$ gives the acceleration reaction of the fluid on $\tilde{D}$.) These remarks serve to motivate, therefore, the introduction of the generalized impulse functional $I(\omega)$ in our variational characterization of steady flows. The existence theory and the asymptotic properties of soutions as $\lambda \to +\infty$ are discussed in detail in [20] along the same lines as the preceding two examples. The classical case first considered by Föppl (see [15, Section 155]) concerns a (symmetric) point vortex pair ($\lambda = +\infty$) in the wake of a circular cylinder ($\tilde{D}$ is a disc); a steady flow is obtained when the point vortex is located at an equilibrium of the appropriate Kirchhoff–Routh Hamiltonian (see [20]).

Other Related Problems

Of the numerous other possible flow geometries of physical interest that can be treated with the same general methods (although they are not necessarily special cases of the problems formulated in 1A), here we shall mention just two. First, we note that (symmetric) vortex streets can be studied much like vortex pairs simply by imposing x-periodic boundary conditions. Such a problem is formulated and analyzed in [18], for instance. Second, we remark that corotating symmetric vortex pairs or, more generally, corotating systems of N vortices with N-fold symmetry can be studied. A complete variational analysis of this type of problem is available in [21]; other literature includes [9, 24].

1C. *Analogous Problems for Axisymmetric Flows*

We now extend the theory of two-dimensional flows outlined above to three-dimensional flows with axisymmetry. We consider an axisymmetric solution of the

Euler equations with velocity field $u = u^z(z, r, t)e_z + u^r(z, r, t)e_r + u^\theta(z, r, t)e_\theta$ and pressure $p = p(z, r, t)$, where (z, r, θ) denotes the usual cylindrical coordinate system and $\{e_z, e_r, e_\theta\}$ is the associated (orthonormal) coordinate frame. The vorticity field, $\omega = \omega(z, r, t)$, is given by

$$\omega = \omega^z e_z + \omega^r e_r + \omega^\theta e_\theta = \frac{1}{r}(ru^\theta)_{,r} e_z - u^\theta_z e_r + (u^r_z - u^z_r)\, e_\theta. \tag{1.17}$$

As is shown in [5, 22], the governing dynamical equations are expressible as a system of evolution equations for the functions

$$\zeta = \frac{1}{r}\omega^\theta, \qquad \gamma = ru^\theta. \tag{1.18}$$

It is convenient to introduce new spatial variables $x = z$, $y = \frac{1}{2}r^2$ for the variable point (x, y) in the cross-section D of the given axisymmetric fluid domain. The Stokes streamfunction $\psi = G\zeta$, G denoting the Green operator as before, is now defined by solving the elliptic boundary value problem,

$$L\psi = \zeta \text{ in } D, \qquad \psi = 0 \text{ on } \partial D,$$

where

$$L = -\frac{1}{2y}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}. \tag{1.19}$$

The equations governing ζ and γ are found to be (after some calculations given in [22])

$$\begin{aligned} \zeta_t + \frac{\partial(\zeta, G\zeta + \bar\psi)}{\partial(x, y)} + \frac{\partial(\gamma/2y, \gamma)}{\partial(x, y)} &= 0 \\ \gamma_t + \frac{\partial(\gamma, G\zeta + \bar\psi)}{\partial(x, y)} &= 0 \qquad \text{in} \quad D \times (0, T), \end{aligned} \tag{1.20}$$

where $\bar\psi = \bar\psi(x, y)$ is the streamfunction for the ambient irrotational flow, $L\bar\psi = 0$. These equations are then the point of departure for our discussion of steady axisymmetric vortex flows.

Vortex Rings without Swirl

When γ is taken to be identically zero, the system (1.20) reduces to a single equation for ζ which has precisely the same form as the two-dimensional vortex dynamics equation (1.2), except that G is the Green operator for L rather than for $-\Delta$. The corresponding theory for steady solutions therefore applies with essentially no changes. This theory is developed in [6] for the model problem in a domain, and in [14] for the case of steadily translating vortex rings (the analog of the vortex pair problem); the existence of solutions ζ_λ and their asymptotic behavior as $\lambda \to +\infty$ are therefore known. One novelty encountered in this case but not in the two-dimensional case is the result that in problems P_2^r and P_2^1 the translational speed $c_\lambda \to +\infty$ as $\lambda \to +\infty$. This makes the formulation of P_2^r and P_2^1 much preferable to that of P_1^r and P_1^1, at least as far as the analysis of properties of the solutions is concerned. Other approaches to the vortex ring problem are taken in [13, 7].

Vortex Rings with Swirl

A new class of steady solutions of (1.20) for which both $\zeta > 0$ and $\gamma > 0$ in a subdomain $\Omega \subset D$ is introduced in the recent work [22]. It is shown therein that there exist solutions in the model case, for instance, having the form (analogous to (1.11))

$$\zeta = \lambda f_s(y, (G\zeta - \mu)^+), \qquad \gamma = \lambda^{1/2} b^{-1}((G\zeta - \mu)^+) \tag{1.21}$$

if the dependence $f(y, s)$ is defined by

$$f(y, s) = [b^{-1}(s)]^2/4y - a(b^{-1}(s)) \qquad (s \geq 0),$$

where a and b are specified functions; these latter functions have physical interpretations as densities for the angular impulse (or momentum) and helicity integrals respectively, as is stressed in [22]. The function $f(y, s)$ is required to [illegible] them here; the so-called Bernoulli case occurs when $a = 0$ and $b(t) = t$, so that $f(y, s) = (s^+)^2/4y$, and the general conditions are fashioned on this special case. The constrained maximization problems analogous to P_1^r and P_2^r that pertain to these solutions are formulated by eliminating γ (algebraically) from the steady equations for the governing system (1.20); we refer the reader to [22] for the full discussion.

2. Iterative Procedures

In this section we define and analyze the iterative procedures which are designed to solve the variational problems formulated in Section 1. We generate these procedures by using a general construction due to Eydeland [10] termed "transformation of the objective functional." Iterative procedures of this kind, which can be regarded as nonlinear analogs to the "power method" for linear eigenvalue problems, have the virtue that their convergence properties can be analyzed even when the given problem does not have a unique solution (and the set of solutions is quite complicated, in principle). In Theorem 2.3, we prove that each version of our procedure converges in a generalized sense independently of its initialization. In

Theorem 2.4; we show that in most cases the procedure behaves as if the solution were unique; we note that this result is supported further by our observations of the implemented computations.

2A. *Analysis in the Case of One Linear Constraint*

We now investigate iterative procedures for solving problems P_1^r and P_1^1 formulated in Section 1. We let K_λ denote the class of functions

$$K_\lambda = \left\{\omega \in L^{r'}(D) : 0 \leqslant \omega \leqslant \lambda,\ C(\omega) = \iint_D \omega\, dx\, dy = 1\right\}, \tag{2.1}$$

where $(\text{meas}(D))^{-1} < \lambda \leqslant +\infty$ and $1/r + 1/r' = 1$. We recall from the definition of these constrained maximization problems that P_1^r is posed on K_∞ while P_1^1 is posed on K_λ $(\lambda < +\infty)$ with $r' = \infty$.

We introduce two versions of the iterative procedure corresponding to the two problems P_1^r and P_1^1. They are defined in an explicit form as follows.

Procedure Π_1^r. Given an arbitrary $\omega^0 \in K_\infty$, let

$$\omega^j = \lambda f_s((G\omega^{j-1} + \bar{\psi} - \mu^j)^+) \in K_\infty \qquad (j = 1, 2, ...), \tag{2.2}$$

where $\mu^j \in \mathbb{R}$ is chosen so that $C(\omega^j) = 1$.

Procedure Π_1^1. Given an arbitrary $\omega^0 \in K_\lambda$, let

$$\omega^j = \lambda \chi_{\{G\omega^{j-1} + \bar{\psi} - \mu^j > 0\}} \in K_\lambda \qquad (j = 1, 2, ...), \tag{2.3}$$

where $\mu^j \in \mathbb{R}$ is chosen so that $C(\omega^j) = 1$.

In both versions the construction of the pair (ω^j, μ^j) from ω^{j-1} can be viewed as a two step process: first, determine $\psi^{j-1} = G\omega^{j-1}$ by solving $-\Delta\psi^{j-1} = \omega^{j-1}$ in D with $\psi^{j-1} = 0$ on ∂D; and, second, find μ^j (and hence ω^j) according to (2.2) or (2.3) so that the constraint $C(\omega^j) = 1$, a nonlinear equation in the parameter μ^j, is satisfied.

For the purposes of analysis it is useful to note the variational form of the above procedures. Procedure Π_1^r can be represented as

$$\omega^j = \arg\max_{\tilde{\omega} \in K_\infty} \iint_D [\tilde{\omega}(G\omega^{j-1} + \bar{\psi}) - \lambda f^*(\tilde{\omega}/\lambda)]\, dx\, dy, \tag{2.4}$$

while procedure Π_1^1 can be represented as

$$\omega^j = \arg\max_{\tilde{\omega} \in K_\lambda} \iint_D \tilde{\omega}(G\omega^{j-1} + \bar{\psi})\, dx\, dy; \tag{2.5}$$

and, for both versions, μ^j is the corresponding Lagrange multiplier. The derivations of the variational conditions (2.2) and (2.3) for the problems (2.4) and (2.5), respectively, follow just as in Section 1 where problems P_1^r and P_1^1 are considered; consequently, we omit them here.

The crucial variational property of this iterative procedure is the monotonicity lemma proved next. In the sequel we use the notation $\|\cdot\|_G$ to denote the G-norm (which is the energy norm)

$$\|\omega\|_G=\left\{\iint_D \omega G\omega \, dx\, dy\right\}^{1/2} \left(=\left\{\iint_D |\nabla G\omega|^2 \, dx\, dy\right\}^{1/2}\right), \tag{2.6}$$

where G is the Green operator.

LEMMA 2.1. (i) *The sequence* $\omega^j \in K_\infty$ *defined by procedure* Π_1^r *satisfies*

$$\tfrac{1}{2}\|\omega^j-\omega^{j-1}\|_G^2 \leqslant \Phi_\lambda(\omega^j)-\Phi_\lambda(\omega^{j-1}) \qquad \text{for} \quad j=1,2,.... \tag{2.7}$$

(ii) *The sequence* $\omega^j \in K_\lambda$ *defined by procedure* Π_1^1 *satisfies*

$$\tfrac{1}{2}\|\omega^j-\omega^{j-1}\|_G^2 \leqslant E(\omega^j)-E(\omega^{j-1}) \qquad \text{for} \quad j=1,2,.... \tag{2.8}$$

Proof. (i) It follows directly from the definition of the objective functional Φ_λ that

$$\begin{aligned}\Phi_\lambda(\omega^j)-\Phi_\lambda(\omega^{j-1}) &= \tfrac{1}{2}\|\omega^j-\omega^{j-1}\|_G^2 \\ &\quad+\iint_D [\omega^j(G\omega^{j-1}+\bar{\psi})-\lambda f^*(\omega^j/\lambda)]\, dx\, dy \\ &\quad-\iint_D [\omega^{j-1}(G\omega^{j-1}+\bar{\psi})-\lambda f^*(\omega^{j-1}/\lambda)]\, dx\, dy \\ &\geqslant \tfrac{1}{2}\|\omega^j-\omega^{j-1}\|_G^2,\end{aligned} \tag{2.9}$$

where the inequality holds by virtue of definition (2.4).

(ii) It follows as in (i) above that

$$\begin{aligned}E(\omega^j)-E(\omega^{j-1}) &= \tfrac{1}{2}\|\omega^j-\omega^{j-1}\|_G^2+\iint_D (\omega^j-\omega^{j-1})(G\omega^{j-1}+\bar{\psi})\, dx\, dy, \\ &\geqslant \tfrac{1}{2}\|\omega^j-\omega^{j-1}\|_G^2,\end{aligned} \tag{2.10}$$

now using definition (2.5).

On the basis of Lemma 2.1 we conclude that (i) the sequence $\Phi_\lambda(\omega^j)$ for Π_1^r and (ii) the sequence $E(\omega^j)$ for Π_1^1 are nondecreasing as $j\to+\infty$. That these sequences are bounded above is easily demonstrated as follows.

(i) We have, using elementary properties of the Green function, the estimate $\sup_G G\omega \leqslant \mathscr{C}_1 \|\omega\|_{L^{r'}}$, where the constant $\mathscr{C}_1$ depends upon r and D. Therefore, we find that for all $\omega \in K_\infty$ there holds

$$\begin{aligned} \Phi_\lambda(\omega) &= \iint_D \left[\tfrac{1}{2}\omega(G\omega + \bar{\psi}) - \lambda f^*(\omega/\lambda)\right] dx\, dy \\ &\leqslant \mathscr{C}_2 \|\omega\|_{L^{r'}} - \mathscr{C}_3 \lambda^{-r'/r} \|\omega\|_{L^{r'}}^{r'}, \end{aligned} \tag{2.11}$$

using estimates for $f^*(\sigma)$ implied by the hypotheses (1.6). Clearly, inequality (2.11) implies that Φ_λ is bounded above on K_∞ when $\lambda < +\infty$ and $1 < r < +\infty$.

(ii) We have (as in (i)) the estimate $\sup G\omega \leqslant \mathscr{C}_4 \|\omega\|_{L^\infty}$, where the constant $\mathscr{C}_4$ depends upon D. Therefore, we find easily that for all $\omega \in K_\lambda$ there holds

$$E(\omega) \leqslant \mathscr{C}_5 \lambda. \tag{2.12}$$

(Sharper estimates than (2.11) and (2.12) can be given, but they are not needed in the present analysis.)

We now conclude that

$$\Phi_\lambda(\omega^j) \uparrow \Phi_\lambda^* < +\infty, \qquad E(\omega^j) \uparrow E^* < +\infty \qquad \text{as } j \to +\infty, \tag{2.13}$$

and hence, by (2.7) and (2.8), that

$$\|\omega^j - \omega^{j-1}\|_G \to 0 \qquad \text{as } j \to +\infty, \tag{2.14}$$

for both iterative procedures Π_1^r and Π_1^1.

Let Ω_1^r denote the set of extremals for the constrained maximization problem P_1^r in K_∞, and let Ω_1^1 denote the corresponding set for P_1^1. In other words, $\omega \in \Omega_1^r$ (resp. Ω_1^1) if and only if $\omega \in K_\infty$ (resp. K_λ) and ω satisfies the variational equation (1.11) (resp. (1.12)). The following lemma constitutes a constructive proof of the existence of these extremals.

LEMMA 2.2. (i) $\Omega_1^r \neq \varnothing$. (ii) $\Omega_1^1 \neq \varnothing$.

Proof of (i). By virtue of (2.11) and (2.13), we have the bound $\|\omega^j\|_{L^{r'}} \leqslant N_0 < +\infty$ for the sequence of iterates ω^j, $j = 0, 1, 2, \ldots$, where N_0 depends upon r, D, λ, and ω^0. Consequently, we can choose a subsequence ω^{j_k} such that

$$\omega^{j_k} \to \omega^* \qquad \text{and} \qquad \omega^{j_k - 1} \to \omega^{**} \qquad \text{weakly in } L^{r'}. \tag{2.15}$$

Using standard properties of the Green operator, we know that

$$G\omega^{j_k} \to G\omega^* \qquad \text{and} \qquad G\omega^{j_k - 1} \to G\omega^{**} \qquad \text{strongly in } L^r. \tag{2.16}$$

The statements (2.15) and (2.16) together with (2.14) imply that $\omega^* = \omega^{**}$, since $\|\omega^* - \omega^{**}\|_G = \lim_{k \to \infty} \|\omega^{j_k} - \omega^{j_k - 1}\|_G = 0$. Now we claim that

$$\omega^* = \arg\max_{\tilde{\omega} \in K_\infty} \iint_D [\tilde{\omega}(G\omega^* + \bar{\psi}) - \lambda f^*(\tilde{\omega}/\lambda)]\, dx\, dy. \tag{2.17}$$

To prove this we recall definition (2.4), which implies that for any $\tilde{\omega} \in K_\infty$,

$$\begin{aligned} &\iint_D [\tilde{\omega}(G\omega^* + \bar{\psi}) - \lambda f^*(\tilde{\omega}/\lambda)]\, dx\, dy \\ &\quad = \lim_{k \to +\infty} \iint_D [\tilde{\omega}(G\omega^{j_k - 1} + \bar{\psi}) - \lambda f^*(\tilde{\omega}/\lambda)]\, dx\, dy \\ &\quad \leqslant \lim_{k \to +\infty} \iint_D [\omega^{j_k}(G\omega^{j_k - 1} + \bar{\psi}) - \lambda f^*(\omega^{j_k}/\lambda)]\, dx\, dy \\ &\quad \leqslant \iint_D [\omega^*(G\omega^* + \bar{\psi}) - \lambda f^*(\omega^*/\lambda)]\, dx\, dy, \end{aligned} \tag{2.18}$$

where we also use the convexity of $f^*(\sigma)$ in the last inequality. Since clearly $\omega^* \in K_\infty$, we obtain (2.17) as claimed. The variational conditions for (2.17),

$$\omega^* = \lambda f_s((G\omega^* + \bar{\psi} - \mu^*)^+) \qquad \text{in } D \tag{2.19}$$

with a Lagrange multiplier $\mu^* \in \mathbb{R}$, follow using the standard argument as sketched in Section 1. (The constant μ^* is uniquely determined by the extremal ω^*.) This proves assertion (i).

Proof of (ii). This proof is analogous to that of (i). Indeed, arguing as before, we now conclude that there is a subsequence ω^{j_k} such that

$$\omega^{j_k}, \omega^{j_k - 1} \to \omega^* \qquad \text{weakly star in } L^\infty, \tag{2.20}$$

$$\omega^* = \arg\max_{\tilde{\omega} \in K_\lambda} \iint \tilde{\omega}(G\omega^* + \bar{\psi})\, dx\, dy. \tag{2.21}$$

The variational conditions for (2.21),

$$\omega^* = \lambda \chi_{\{G\omega^* + \bar{\psi} - \mu^* > 0\}} \qquad \text{in } D, \tag{2.22}$$

follow again as sketched in Section 1. This proves assertion (ii).

The above proof of Lemma 2.2 also establishes that every subsequence of iterates ω^{j_k} has a further subsequence $\omega^{j'_k}$ which converges in the G-norm to an extremal ω^*; this holds both for Π_1^r and Π_1^1. Consequently, if we let $\mathrm{dist}_G(\omega, \Omega) = \inf_{\tilde{\omega} \in \Omega} \|\omega - \tilde{\omega}\|$ denote the G-norm distance to a set Ω, then we obtain the following theorem which is our generalized convergence result.

THEOREM 2.3. (i) *For procedure* Π_1^r *defined by* (2.2), *we have*

[redacted]

$$\operatorname{dist}_G(\omega^j, \Omega_1^1) \to 0 \qquad as \quad j \to +\infty. \tag{2.24}$$

Proof. The proofs of (i) and (ii) are now immediate. Arguing by contradiction, we see that it is impossible that $\operatorname{dist}_G(\omega^{j_k}, \Omega_1^r) \geqslant \delta > 0$ for any subsequence ω^{j_k} of ω^j $(1 \leqslant r < +\infty)$.

We cannot conclude from Theorem 2.3 that the sequence ω^j is convergent (or Cauchy) in the G-norm when Ω_1^r $(1 \leqslant r < +\infty)$ consists of more than one extremal. In the next theorem, however, we establish an alternative which ensures that this lack of convergence must be quite exceptional. Let A_1^r $(1 \leqslant r < +\infty)$ be the set of G-norm limit points of the iterative sequence ω^j for a given initialization ω^0. (Of course, $A_1^r \subseteq \Omega_1^r$ by Lemma 2.2.) We characterize the convergence properties of the procedures Π_1^r $(1 \leqslant r < +\infty)$ in terms of the nature of the sets A_1^r.

THEOREM 2.4. *For both* (i) $1 < r < +\infty$ *and* (ii) $r = 1$, *we have the alternative: either* A_1^r *consists of a single extremal, or* A_1^r *contains infinitely many extremals none of which is isolated* (*in the G-norm*).

Proof. We use the proof suggested in [11]. We claim that if A_1^r contains one isolated extremal ω^*, then the (entire) sequence ω^j converges to ω^* as $j \to +\infty$. To prove this, we take disjoint neighborhoods N_1 and N_2 of ω^* and $A_1^r \backslash \{\omega^*\}$, respectively, with $\delta = \operatorname{dist}_G(N_1, N_2) > 0$. By the reasoning used in Theorem 2.3, $\omega^j \in N_1 \cup N_2$ for all sufficiently large j. But, by (2.14), $\|\omega^j - \omega^{j-1}\|_G \leqslant \delta/2$ for all sufficiently large j. Consequently, as $\omega^* \in A_1^r$, there is a j_0 such that $\omega^j \in N_1$ for all $j \geqslant j_0$, and the claim follows. The stated alternative is now evident, and so the theorem is proved.

We remark that stronger versions of Theorems 2.3 and 2.4 can be proved in which the G-norm is replaced by (i) the $L^{r'}$-norm when $1 < r < +\infty$, or (ii) the L^q-norm for any $q < +\infty$ when $r = 1$. These extensions of the convergence results can be demonstrated by using a standard bootstrap argument.

Finally, we remark on the determination of μ^j in (2.4) and (2.5). We note that if

(i) $\omega(\mu) := \lambda f_s((G\omega^{j-1} + \bar{\psi} - \mu)^+)$ or

(ii) $\omega(\mu) := \lambda \chi_{\{G\omega^{j-1} + \bar{\psi} - \mu > 0\}}$

then the circulation $C(\omega(\mu))$ is a monotonic function of μ, and thus $\mu = \mu^j$, satisfying $C(\omega(\mu^j)) = 1$, can be found by a simple method such as a binary search procedure. Below in 2B we consider a general approach to finding these constants.

2B. *Analysis in the Case of Two Linear Constraints*

The following two versions of the iterative procedure for the two problems P_2^r and P_2^1 are obvious analogs to those studied in the preceding subsection.

Procedure Π_2^r. Given an arbitrary $\omega^0 \in K_\infty$ with $I(\omega^0) = m$, let

$$\omega^j = \lambda f_s((G\omega^{j-1} - c^j\eta - \mu^j)^+) \in K_\infty \qquad (j = 1, 2, ...), \tag{2.24}$$

where $c^j, \mu^j \in \mathbb{R}$ are chosen so that $I(\omega^j) = m$, $C(\omega^j) = 1$.

Procedure Π_2^1. Given an arbitrary $\omega^0 \in K_\lambda$ with $I(\omega^0) = m$, let

$$\omega^j = \lambda \chi_{\{G\omega^{j-1} - c^j\eta - \mu^j > 0\}} \in K_\lambda \qquad (j = 1, 2, ...), \tag{2.25}$$

where $c^j, \mu^j \in \mathbb{R}$ are chosen so that $I(\omega^j) = m$, $C(\omega^j) = 1$.

As is entirely evident, we can prove the same results concerning the convergence properties of Π_2^r and Π_2^1 as we have proved in subsection 2A for Π_1^r and Π_1^1. We will not state the corresponding theorems here. (In fact, these results are valid even for any number of linear constraints.)

The only difference between the procedures Π_1^r and Π_2^r $(1 \leq r < +\infty)$ stems from the fact that the latter requires the determination of two constants c^j, μ^j at each iteration j. Consequently, unlike the former case, we cannot use a simple binary search procedure to compute these constants. We now propose a general method for finding the pair c^j, μ^j in (2.24) and (2.25). (This method also applies to any number of linear constraints.)

We consider first the procedure Π_2^r. It is clear that c^j, μ^j are the solutions of the following system of nonlinear equations with respect to the variables c, μ:

$$\begin{aligned} &\iint_D \lambda f_s((G\omega^{j-1} - c\eta - \mu)^+)\, dx\, dy = 1 \\ &\iint_D \eta\lambda f_s((G\omega^{j-1} - c\eta - \mu)^+)\, dx\, dy = m. \end{aligned} \tag{2.26}$$

But this system can be written in the equivalent form $\nabla R^j(c, \mu) = 0$, where

$$R^j(c, \mu) = \iint_D \lambda f((G\omega^{j-1} - c\eta - \mu)^+)\, dx\, dy + mc + \mu. \tag{2.27}$$

The function $R^j(c, \mu)$ is convex by hypothesis (1.6). Thus, the determination of c^j, μ^j is reduced to the minimization of $R^j(c, \mu)$, which can be accomplished by a standard method such as the steepest descent method.

We consider second the procedure Π_2^1, for which the problem of finding c^j, μ^j is completely analogous. Now it is necessary to minimize the convex function

$$R^j(c, \mu) = \iint_D \lambda(G\omega^{j-1} - c\eta - \mu)^+\, dx\, dy + mc + \mu. \tag{2.28}$$

3. Numerical Implementation

We begin this section with a description of the discretization of the iterative procedures defined in Section 2. We then present some numerical results for two special problems chosen to illustrate the general method.

3A. *Discretized Iterative Procedures*

In the numerical implementation of the procedures Π_1^r and Π_2^r $(1 \leqslant r < +\infty)$ we use a finite-element method. We note, however, that other methods are possible and that the choice of a particular discretization is important only as far as the solver for the linear elliptic problem (3.1) is concerned. We shall discuss the discrete version of Π_1^r $(1 < r < +\infty)$ only, since the corresponding discussions of the other procedures are similar.

We let $\mathcal{T}_h$ be a triangulation of the (polygonal) domain D where $h = \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$, T denoting any triangle in the triangulation $\mathcal{T}_h$. (Throughout this discussion, h will be used to indicate a relation to the discrete implementation.) We use a conforming finite-element method, and thus we choose a finite dimensional space V_h of piecewise Lagrange polynomials (for instance, piecewise linear functions) associated with the triangulation $\mathcal{T}_h$ such that $V_h \subset H^1(D) \cap C^0(D)$ and $v_h|_{\partial D} = 0$ for every $v_h \in V_h$. We let π_h denote the projection of $H_0^1(D)$ onto V_h, so that $\pi_h v$ is the V_h-interpolant of the function $v \in H_0^1(D)$. The discretized Green operator $G_h: L^2(D) \to V_h$ is defined in this context as follows: $\psi_n = G_h \omega$ is the finite-element solution of the problem (1.1) or, more precisely, of the problem

$$\iint_D \nabla\psi_h \cdot \nabla\phi_h \, dx\, dy = \iint_D \omega\phi_h \, dx\, dy \qquad \text{for all} \quad \phi_h \in V_h. \tag{3.1}$$

We also write $\bar{\psi}_h = \pi_h \bar{\psi}$ throughout this discussion. With these notations we can now define the finite-element version of the iterative procedure Π_1^r.

Procedure $\Pi_{1,h}^r$. Given an arbitrary $\omega_h^0 \in V_h$ satisfying $\omega_h^0 \geqslant 0$ and $\iint_D \omega_h^0 \, dx\, dy = 1$, let

$$\omega_h^j = \lambda\pi_h f_s((G_h\omega_h^{j-1} + \bar{\psi}_h - \mu_h^j)^+) \qquad (j = 1, 2, ...), \tag{3.2}$$

where $\mu_h^j \in \mathbb{R}$ is chosen so that $\iint_D \omega_h^j \, dx\, dy = 1$.

We note that μ_h^j is uniquely determined (by ω_h^{j-1}) since the expression

$$\iint_D \lambda\pi_h f_s((G_h\omega_h^{j-1} + \bar{\psi}_h - \mu_h)^+) \, dx\, dy$$

is clearly a monotonic function of μ_h.

As in Section 2, we also have a variational characterization of the procedure $\Pi^r_{1,h}$:

$$\omega_h^j = \arg\max_{\tilde{\omega}_h \in K_{\infty,h}} \iint_D [\tilde{\omega}_h(G_h\omega_h^{j-1} + \bar{\psi}_h) - \lambda\pi_h f^*(\tilde{\omega}_h/\lambda)]\, dx\, dy, \tag{3.3}$$

where $K_{\infty,h} = \{\tilde{\omega}_h \in V_h : \tilde{\omega}_h \geq 0, \iint_D \tilde{\omega}_h\, dx\, dy = 1\}$.

In order to prove that (3.2) follows from (3.3) we observe that $\omega_h \in V_h$ is uniquely determined by its values at the interpolation nodes (a_n, b_n), $n = 1, ..., N$, for $\mathscr{T}_h$, and that

$$\iint_D [\tilde{\omega}_h(G_h\omega_h^{j-1} + \bar{\psi}_h) - \lambda\pi_h f^*(\tilde{\omega}_h/\lambda)]\, dx\, dy$$

$$= \sum_{n=1}^{N} \alpha_n [\tilde{\omega}_h(a_n, b_n)(G_h\omega^{j-1}(a_n, b_n) + \bar{\psi}_h(a_n, b_n)) - \lambda f^*(\tilde{\omega}_h(a_n, b_n)/\lambda]$$

$$\iint_D \tilde{\omega}_h\, dx\, dy = \sum_{n=1}^{N} \alpha_n \tilde{\omega}_h(a_n, b_n),$$

replacing the integrals with quadrature formulas and using the obvious equality $\pi_h f^*(\tilde{\omega}_h(a_n, b_n)/\lambda) = f^*(\tilde{\omega}_h(a_n, b_n)/\lambda)$. Then the variational conditions (3.2) for the finite-dimensional constrained maximization problem (3.3) follows as in Section 2. Now, moreover, we may proceed as in Section 2 to prove that

$$\text{dist}_{G_h}(\omega_h^j, \Omega^r_{1,h}) \to 0 \qquad \text{as} \quad j \to +\infty, \tag{3.4}$$

where $\Omega^r_{1,h} \subseteq V_h$ is the set of solutions of

$$\omega_h = \lambda\pi_h f_s((G_h\omega_h + \bar{\psi}_h - \mu_h)^+) \qquad \text{with} \quad \iint_D \omega_h\, dx\, dy = 1. \tag{3.5}$$

Finally, we complete this discussion by establishing that

$$\sup_{\omega_h \in \Omega^r_{1,h}} \text{dist}_G(\omega_h, \Omega^r_1) \to 0 \qquad \text{as} \quad h \to 0, \tag{3.6}$$

where Ω^r_1 denotes the set of solutions of problem P^r_1. Indeed, if on the contrary there exists a sequence ω_{h_m} such that $h_m \to 0$ and $\text{dist}_G(\omega_{h_m}, \Omega^r_1) \geq \delta > 0$ as $m \to \infty$, then, arguing as in Lemmas 2.2 and 2.3, there exists a further subsequence (denoted again by ω_{h_m}) such that $\omega_{h_m} \to \omega^*$ in the G-norm. Since the multipliers μ_h are bounded for all h (as is easily demonstrated), we may assume that $\mu_{h_m} \to \mu^* \in \mathbb{R}$, taking a subsequence if necessary. Now, we pass to the limit in (3.5) using the standard estimate from finite-element theory, $\|1 - \pi_h\|_{H^1} = O(h)$ for regular families of triangulation [8], and the fact that $G_{h_m}\omega_{h_m} \to G\omega^*$ as $m \to \infty$, and we obtain that

$$\omega^* = \lambda f_s((G\omega^* + \bar{\psi} - \mu^*)^+) \qquad \text{with} \quad \iint_D \omega^*\, dx\, dy = 1.$$

Thus, $\omega^* \in \Omega^r_1$ and so we have the required contradiction.

Procedure $\Pi^1_{1,h}$ $(r=1)$ is defined in the same way as above except that the iterates ω^j belong to the finite-dimensional space of piecewise constant functions associated with the triangulation $\mathscr{T}_h$. The corresponding procedures $\Pi^r_{2,h}$ $(1 \leq r < +\infty)$ with two linear constraints are entirely analogous.

3B. *Computational Examples*

We consider first the constrained impulse version, P^r_2, of the symmetric vortex pair problem formulated in Section 1B. For these flows the fluid domain is taken to be $D=\{-1<x<1,\ 0<y<1\}$, and the constraints $C(\omega)=1$, $I(\omega)=0.25$ are prescribed. In Fig. 1, we present the streamline patterns for these steadily translating vortex pairs with $\lambda=20$ and $f(s)=(s^+)^r/r$, where we take successively the values (a) $r=1$, (b) $r=1.25$, (c) $r=1.5$; we indicate the free-boundary as a solid curve, and all other streamlines as dashed curves. The computed values of the constants c and μ associated with the solution in each case are: (a) $c=0.27$, $\mu=0.12$; (b) $c=0.24$, $\mu=0.07$; (c) $c=0.23$, $\mu=0.02$. In qualitative terms, the displayed solutions illustrate the dependence of the size and shape of a steady vortex (with fixed circulation and impulse) upon the distribution of vorticity within its core. Indeed, the vortex core broadens considerably as the exponent r is increased, and hence as the vorticity-streamfunction relation $\omega=\lambda[(\psi-cy-\mu)^+]^{r-1}$ is changed;

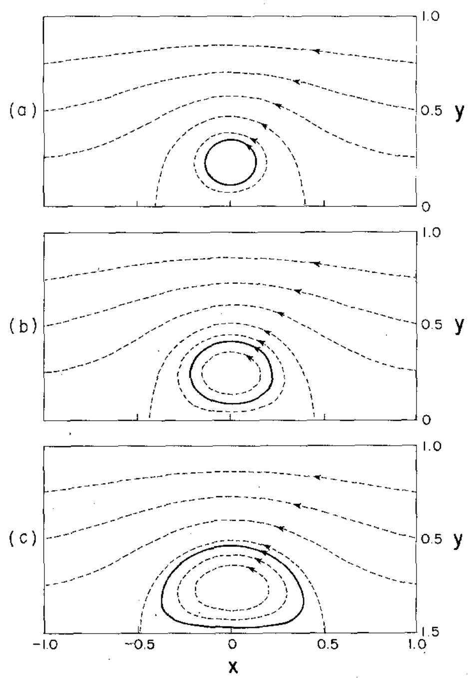

FIGURE 1

in the nearly extreme case (c), the free-boundary almost coincides with the separating streamline $\psi - cy = 0$. These phenomena are associated with the decrease of μ to zero. The translation speed c, on the other hand, changes only slightly over this range of r values.

We consider second the constrained impulse version, P_2^1, of the Föppl-type vortex wake problem also discussed in Section 1B. Now the domain is taken to be $D = \{-1.25 < x < 0.75,\ 0 < y < 1,\ \sqrt{x^2 + y^2} > 0.25\}$, and the constraints $C(\omega) = 1$, $I(\omega) = m$ are prescribed in the three cases: (a) $m = 0.1$; (b) $m = 0.2$; (c) $m = 0.3$. In Fig. 2, we give the streamline patterns for these vortical wakes behind a circular cylinder with $f(s) = s^+$ where again $\lambda = 20$. The computed values of the constants c and μ are: (a) $c = 0.73$, $\mu = 0.02$; (b) $c = 0.37$, $\mu = 0.09$; (c) $c = 0.14$, $\mu = 0.18$. As is expected on the basis of the classical point vortex model due to Föppl [15, 20], the vortex (core) moves away from the obstacle (the cylinder) as the impulse m is increased; in this process, c decreases and μ increases. (We note that in case (c) the vortex is already far enough away from the obstacle that the flow is measurably affected by our domain truncation, and so it would be expected to depart from the actual flow in the exterior of the obstacle.) We remark that the constrained impulse version of the numerical method is especially convenient in this context because it avoids the ad hoc adjustments which are necessary to obtain the desired wake if the value of the speed c is prescribed instead.

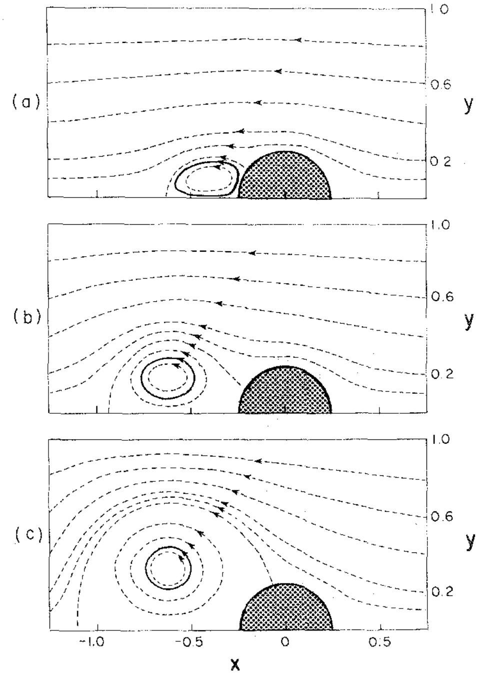

FIGURE 2

In all of the computations discussed above, the discrete grid consisted of 3321 nodes with $h = 0.025$. The iterative procedure $\Pi^r_{2,h}$ was terminated at iteration j when

$$\max\{\|\omega_h^j - \omega_h^{j-1}\|_{L^1}/\|\omega_h^j\|_{L^1},\ \|\omega_h^j - \omega_h^{j-1}\|_G/\|\omega_h^j\|_G\} \leqslant 10^{-3}.$$

With this stopping criterion, between 20 and 30 iterations were required to compute the displayed solutions. A linear rate of convergence was consistently exhibited by these computations over the range of prescribed parameters and initializations. This supports the practicality of the general method discussed in Section 2, and furthermore indicates that the possible anomalies in the solution sets treated in Section 2 are indeed quite rare.

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